

Platonic Solids and Their Symmetries

Anders Sandberg

Some Random Thoughts about the Occult Correspondences of the Platonic Solids and Their Symmetries

Anders Sandberg (nv91-asa@eufrat.nada.kth.se) 14 Dec 1994 writes:

Very little of modern mathematics has been used in the Cabala, which relies mainly on simple arithmetic operations and some basic combinatorics (an area which it in fact partially founded). I think this is regrettable, since there is a plethora of interesting mathematical results which could be applied to occultism. In the following I will discuss a few interesting areas of solid geometry and abstract algebra. The discussion will be rather non-mathematical, and I will not attempt stringency, which anyway is a bit hard to apply when discussing occult matters.

It is helpful to have models or good renditions of the various polyhedra available to visualise the various properties I will discuss below (like [RW] or [C]), since I cannot include pictures of the often quite complex structures.

The Five Platonic Solids

The Platonic solids, also known as the regular polyhedrons, are the three-dimensional bodies whose surfaces consist of identical, regular polygons which meet in equal angles at the corners. There are five such polyhedrons, the Tetrahedron, the Octahedron, the Cube, the Icosahedron and the Dodecahedron. The first three have apparently been known since ancient times. The others was definitely known by the Pythagoreans, since one of them, Timaeus of Locri, invented the "Platonic" correspondence between them and the elements. Plato later publicised their results, which is the reason they bear his name. Here is a table with their properties (based on [C]):

	Faces Edges	Vertices	Schfli	Dual	Plato	symbol
Tetrahedron	4	6	4	{3,3}	Tetrahedron	Fire
Octahedron	8	12	5	{3,4}	Cube	Air
Cube	6	12	8	{4,3}	Octahedron	Earth
Icosahedron	20	30	12	{3,5}	Dodecahedron	Spirit
Dodecahedron	12	30	20	{5,3}	Icosahedron	Water

[The Schfli symbol represents the type of polygons making up the faces and the number which meet at each vertex. A cube consists of squares (4) and three squares meet at each corner (3), thus its symbol is {4,3}]

[Two polyhedra are duals if the vertices of one correspond one-to-one to the centres of the faces of the other.]

The Platonic Correspondences are Tetrahedron: Fire, Icosahedron: Water, Octahedron: Air, Cube: Earth and Dodecahedron: The Quinta Essentia. While this is pleasing from a traditional and aesthetic standpoint, I have not found it workable from a more mathemagickal standpoint.

These solids naturally fall into three groups, based on their symmetries and duals. The Octahedron and Cube, which are duals of each other, form one group, while the Dodecahedron and Icosahedron form another. The Tetrahedron form a third group with only itself as a member since it is its own dual. Note that the five elements are similarly divided: the spiritual elements are duals to the material elements (and a similar duality holds for actives and passives), and the fifth is left out or its own opposite (one is reminded of the concept of positive and negative aethyr in [CL]). Thus, from my mathemagickal standpoint, Quintessence belongs more naturally to the Tetrahedron, the Cube and Octahedron corresponds as normal to Earth and Air while Fire and Water correspond to the Dodecahedron and Icosahedron respectively. I will now discuss the properties of the various polyhedrons from different perspectives.

The Tetrahedron

The Tetrahedron classically represents Fire, and each face is also the alchemical triangle of fire. The Golden Dawn called it the Pyramid of Fire, and used it as the admission badge for the path of Shin. The three upper triangles represents Solar Fire, Volcanic Fire and Astral Fire, while the bottom triangle, often hidden from view is the latent heat. The upper triangles are also linked to the three fire-signs Aries, Saggittarius and Leo.

Note that each face and each vertex can be put into a one-to-one correspondence with an element. Each element touches the others, showing that the superficial divisions of Fire and Water, Air and Earth are really unities. No element is superior to any other, and they all balance each other into a very stable structure (Buckminster Fuller designed his entire mathematics and architecture on this simple fact). This represents is in my view the state before the divisions between the elements, and thus resonant with the Quinta Essentia, from which the element were formed.

Its worth noting that the tetrahedron is its own dual. At the same time it belongs to the 4.3.2 symmetry group, the same as the octahedron and the cube belongs to. In a way this reflects "Keter is in Malkuth, and Malkuth is in Keter", the material world subtly reflects the spiritual world and vice versa.

The four elements are linked with six edges, which may correspond to the hexagrams and the planets (the Sun is as usual in the centre). Seeing things this way, each planet can be seen as a path between two elements. Some possible correspondences (this probably requires more thought, and I would be happy to hear other possibilities):

Moon Water-Fire Mercury Air-Fire Venus Water-Earth Mars Fire-Earth Jupiter Air-Water Saturn Earth-Air

We will see that this planetary/double-element correspondence is common in the other structures too, making it very interesting.

The Octahedron

The Octahedron corresponds classically to Air. It has 8 faces (corresponding to Hod and mental activity?), 6 vertices and 12 edges. The edges naturally correspond to the zodiac. They can be arranged in such a manner that the four triplicities border a triangular face each without overlap. These faces cover half the surface, leaving 4 incomplete faces with signs from three elements along each edge (this may signify an absence of the left-out element. The octahedron thus consists of both the abundance of each element and its absence). At each corner two elements meet (creating the same planetary correspondences as in the tetrahedron, with the sun at the centre as usual). In this arrangement, each square "equator" corresponds to one quadruplicity.

Another common use of the octahedral symmetry is used in banishing rituals (mainly the LBRP and the Rose-Cross Rite). The sphere encircled by three orthogonal circles is the natural projection of the octahedron onto the surface of a sphere. In most rituals the horizontal equator corresponds to the cherubic signs. This also corresponds to the six directions of the Yetziratic Sealing Rite [DK], see below for the discussion of the symmetric group.

The octahedron fits air very well, since the various symmetries and correspondences are so clear and easily viewed. As we will see in the case of the cube, many of these symmetries are hidden or hard to discern in the case of Earth, perhaps signifying that the intellect allows us to see the structure of the world more easily than our physical senses, which are parts of the system we try to study.

The Cube

The Cube naturally corresponds to Earth. It is stable, the basis of western architecture and salt crystallises into cubes. It has six faces, making some groups attribute it to Tiphareth. The six faces naturally fit the sephira, and can of course be linked to the planets except for the sun, which is placed in the centre. Another natural link is the folded out cube, which forms a cross.

The eight corners of the cube neatly corresponds to three complementary dualities. When two dualities interact, the four elements are created. Now the four elements are dualized again, and we get eight corners representing the relative absence and abundance the each element. This is naturally dual to the faces of the octahedron. In the same way the six faces correspond to the six vertices of the octahedron (i.e. meetings between two quadruplicities). It is however not possible to arrange the three quadruplicities along the edges to enclose whole faces without overlaps. Does this signify the imperfections and limitations of the material world?

Its an interesting fact that the cube isn't stable. If a model is made using toothpicks and peas, it can easily be shown that it tends to distort or collapse. It is however possible to inscribe a tetrahedron inside a cube so that its vertices meet four corners of the cube and its edges lie in the faces of the cube. This will stabilise it completely (spirit stabilises and orders matter). If two tetrahedrons are inscribed using different sets of vertices, they

intersect and form a geometric body known as the "Stella Octangula" (which is an octahedron with pyramids added on its faces). This is a very neat representation of the complementarity between positive and negative forces, which seems to underlie much of the structure of the cube.

It is worth noting that the duality of the cube and octahedron fits the duality between Air and Earth. Both belong to the same symmetry family (called 4.3.2), to which all normal minerals and crystals belong (only the so-called quasicrystals belong to the icosahedral symmetry family). It is also an interesting fact that of the platonic polyhedrons, only the cube can fill space completely, without interstices or overlaps. Thus we see that despite that the only way to create a completely consistent universe out of one element is to use matter. The other elements are not able to bind together in the right way to form a stable world, but will either move around or form imperfect patterns.

The Icosahedron

This polyhedron traditionally corresponds to water, possibly because it rolls quite easily. Its 20 faces could correspond to the sephiroth and qliphoth, but I have so far not found any significant arrangement. While the octahedron and cube, belonging to 4.3.2 have many symmetries involving the four elements, trinities and dualities, the icosahedron and dodecahedron, belonging to 5.3.2 have links to the five elements and the trinities and dualities. Thus they correspond closer to the whole system than the more material elements, which deal with just the four elements.

In nature these symmetries are rare, and are usually found in viruses and radiolaria. One reason for the rarity of these symmetries may be that they don't interconnect as well as the 4.3.2 group. In crystals, molecules and viruses with 5.3.2 symmetries organize according to the 4.3.2 group instead, subjugating their own symmetries. The higher elements decay into the lower in order to form the world.

These symmetries are harder to discern, since traditionally we humans have a tendency to avoid high-order groups, especially odd symmetries (its worth noting that the number five is sacred to the Discordians since it is the smallest number of factors the human mind is unable to handle at once).

The 12 vertices can of course be viewed as the zodiac. In this case each sign is linked to five other signs along the edges which corresponds to the five elements, a quite interesting set of correspondences (this is of course reflected in the faces and edges of the dodecahedron in a similar way). This seems to imply a network between the signs, where each sign is transformed into five others by the actions of the five elements. I have so far not seen any uses for this system, but it is potentially interesting.

One obvious way of arranging the elements in such a pattern is the following: choose two edges opposite to each other and assign them to an element. Then there are four edges along the "equator" if the two edges are regarded as the poles which can be assigned the same element. These edges are orthogonal to the first, and each pair of opposite edges are orthogonal to all others. In fact, if the opposite edges are joined with lines through the interior, a very neat structure of interlocking rectangles result, where each rectangle locks the other rectangles without touching them. Each pair of rectangles

doesn't interlock, but together they form a synergetic whole. In this way each element can be assigned to its own edges in a proper way. It is interesting to note that the pattern inside each element belongs to the 4.3.2 symmetry group.

The icosahedron can be inscribed in the octahedron if its vertices are placed on the octahedron-edges in the golden ratio. In this case eight faces of the icosahedron lie in the plane of the faces of the octahedron, and the rest lie in the interior. As a general rule, the golden ratio is intimately linked to the 5.3.2 family of solids. This construction is symbolic of how the creativity and feeling of Water is needed to form the rational thought of Air.

In my own system the icosahedron corresponds to water. It seems to tie together things in complex, apparently random ways and encompass them without necessarily elucidate their interrelationships. As one can see, the complexity of the icosahedron and dodecahedron "liquiefies" the various correspondences. The number of possible arrangement is much larger than for the relatively simple cubes and octahedrons.

The Dodecahedron

This solid is classically attributed to spirit, probably because it was the last discovered and because of the pentagonal faces. Its twelve faces has naturally been attributed to the zodiac, and there have even been dodecahedral calendars. The symmetries discussed above exist in a dual form here too.

The dodecahedron can be seen as the union of five intersecting cubes, whose corners touch the vertices of the dodecahedron (this is a rather complex structure and hard to visualize). At each vertex three different cubes meet. Along each side of the dodecahedron an edge from a cube runs, creating a rather neat system of correspondences between the five elements and the edges like the system mentioned above for the icosahedron.

Another way of placing polyhedrons in the dodecahedron is to use five intersecting tetrahedrons, whose corners touch the vertices. This is a most elegant configuration where the tetrahedrons seem to twist around each other. It exists in two different forms, essentially corresponding to clockwise and counterclockwise rotation. The space occupied by all five tetrahedons is a smaller icosahedron, another nice example of the power of duals. It could perhaps be seen as a "construction drawing" of Fire, where the Quinta Essentia takes on its various elemental properties, and combines them in an eternally rotating and twisting form.

The evolution of the Quinta Essentia into the four elements may thus be described as follows: The original form of the Tetrahedron is created out of the primordial chaos by being the simplest and most stable form. It combines in various ways with itself, either by moving and mixing, forming the Dodecahedron and Fire, or by linking together and building the Icosahedron and primordial Water. However, while both polyhedrons are close to being perfect spheres, they don't fit together. These imperfect interactions between the growing numbers of polyhedrons force them to order themselves according to cubical symmetries, and Earth and Air are formed. As we will see, this fits with some results within group theory.

Before we shift our focus to the abstract properties of groups, its worth mentioning that there exist other polyhedrons of potential magickal interest.

Other Polyhedra

One such set is the Kepler-Poinsot polyhedra, also known as star polyhedra. They are a generalisation of the platonic solids, where faces no longer have to be normal polygons but can be star-polygons (like pentagrams) instead, and they may intersect. The four star polyhedrons are called the small stellated dodecahedron, the great stellated dodecahedron, the great dodecahedron and the great icosahedron. They all belong to the same symmetry group as the dodecahedron and icosahedron. These fascinating polyhedrons can be seen to correspond to the four elements. The small stellated dodecahedron has pentagrams as faces. The great dodecahedron, its dual, has intersecting pentagonal faces. The great stellated dodecahedron has also pentagram faces and its dual, the great icosahedron has triangular faces.

I would say the great icosahedron corresponds to Fire (its faces are triangles, and it opens almost like an erupting flower). Its dual, the great stellated dodecahedron correspond to Water. The small stellated dodecahedron, with its pentagonal pyramids rising from the pentagrammal faces correspond to Air, and the almost asteroidlike (it looks a bit like a sphere with dents) great dodecahedron as Earth. Like I mentioned above, the {5.3.2} group seems to exist on a higher level (Perhaps in Briah if the {4.3.2} group exists in Yetzira), containing the symbolism and patterns of the other group in abstract form.

Beside these polyhedrons, there are the Archimedean polyhedrons. These are polyhedrons where the faces can be different regular polygons (no intersections or star-faces allowed). There are 14 of these, with no known occult connotations. This is an area where much further discovery is possible.

The Magick of Groups

Groups are among the most useful mathematical concepts, and can be readily applied to magick (what cannot be applied?). A Group is a set with an associated binary operation $*$ on the set, with the following three axioms:

1. There is an element e in the set such that $e*x=x*e=x$ for all x .
2. The operation $*$ is associative $(a*b)*c=a*(b*c)$.
3. There exists an element x' for every x in the group, such that $x'*x=x*x'=e$.

Note that these three axioms fit quite well with the three supernal sepiroth. However, the $*$ operation seldom seems to have any obvious occult interpretation.

The simplest group (except the unit group with just a unit element) is the cyclic 2-group Z_2 . It has two elements which correspond to the two sides of a duality. The next simplest is of course the cyclic 3-group with three elements, corresponding to a trinity.

The most important group in algebraic magick is of course the group of the four elements. However, there are two groups with four elements, the cyclic group Z_4 and the Klein 4-group. The respective multiplication tables are:

	Z_4				Klein			
	e	a	b	c	e	a	b	c
e	e	a	b	c	e	a	b	c
a	a	b	c	e	a	e	c	b
b	b	c	e	a	b	c	e	a
c	c	e	a	b	c	b	a	e

I use e= Earth, a=Air, b=Water c=Fire.

To which group does the four element belong? This depends a bit on your perspective on the elements. If one sees them as a cyclic organisation, where each element is succeeded by the next, Z_4 is a natural choice. However, if one carefully studies the Klein group, one sees that it consists of two parts. The material elements form a subgroup, where the interactions between Air and Earth form only themselves. When the astral elements are added, the interactions between themselves also form the material elements (the descent from the astral to the material plane). However, when they interact with matter, it can be elevated to the astral level. The group is also the cartesian product of two 2-groups, which fits in well with the division between actives/passives and astrals/materials. This is the reason I think the group fits best for the elements.

If the elements correspond to the Klein 4-group, what does the cyclic group correspond to? If one studies the quadruplicities in the literature, one quickly find that practically all sets of four symbols correspond to the four elements. However, there seems to be one important quadruplicity which fits the cyclic group, the INRI formula. It denotes a linear progression, but at the same time the first and last step are the same. In terms of group theory and the Golden Dawn system, one could say that the first Yud represents the stable, unenlightened state. Nun, death and destruction, forces a change which leads to Resh, rebirth and light which becomes the second Yud, representing the relatively enlightened state where the process can begin anew ("Every day is an initiation").

Multiplication table:

	I1	N	R	I2
I1	I1	N	R	I2
N	N	R	I2	I1
R	R	I2	I1	N
I2	I2	I1	N	R

[I1 and I2 represents the first and second yud, respectively]

Its worth noting that the orbit of Nun (the elements x, x^2, x^3, \dots) generates the whole group, it goes forward all the time. The orbit of the first Yud is simply itself, the

unenlightened state cannot change without any external stimuli. Resh, the sun, has an orbit spanning itself and the first yud. Light in itself can only become no light or more light, not something else. However, the second yud has an orbit which moves backwards along the sequence, ending up at the first yud and then continuing around to itself. The enlightened are able to move as they want, and have no fear of light, death or being unenlightened.

If we look at the orbits of the Klein group in the same way, we find that each element has an orbit consisting of itself and the unity element earth; a pure element can become material, but not create anything else. A mixture of element is necessary to create the whole universe. However, it is necessary only to start with two elements to create it; fire and water can produce earth and air (but not vice versa).

There are just one group of order five, the cyclic 5-group Z_5 since the only groups of prime orders are cyclic (this follows from the very useful theorem of Lagrange which states that the orders of subgroups must divide the order of the group). It is interesting to see that the elements in themselves might form a non-cyclic group, but when Spirit is added, the result is cyclic. Fire emanates from spirit, and is then in turn transformed through Water, Air and Earth until it returns to its source.

The orbits of the Z_5 group naturally lead to the theory of lineal figures (pentagrams, hexagrams etc.). A cyclic group can be generated by an element in different ways, depending on its size. Z_5 can be generated in essentially two ways: by going through each of its elements in turn (1234512345123...) or by going through every two elements (135241352413...). This corresponds to the pentagon and pentagram respectively. It can be shown that for a cyclic group of order n , every choice of a "step length" k which is relatively prime to n creates a lineal figure (other choices doesn't generate the whole group). For example, the cyclic group of order 7 can be generated by steps of length 1 (the heptagon), 2 (the "even heptagram") and 3 (the "spiky heptagram"). The other choices will just create these three figures. For the cyclic group of order 8 there are just the octagon and octagram (stepsize 3 or 5), the steps of size 2,4 and 6 produces just two nested squares. For further correspondences of these figures, see the paper on lineal figures in [IR].

The most important group of order six is the permutation group of three elements (also known as the symmetric group of three letters), S_3 . It consists of the six possible permutations of three letters. These permutations are mentioned in the Sepher Yetzirah (which in fact is a precursor to the study of permutation groups), where the 6 permutations of the three different letters Yud, Heh and Vau of the Name were used to create the directions (this is also used in the Yetziratic Sealing Rite). These permutations fit with the corners of an octahedron or the faces of a cube.

Such symmetric groups are very important, since it can be shown (Cayley's Theorem) that all groups are isomorphic to a group of permutations (which are subgroups of the symmetric groups). This means that if we have a group (say the group of elements), we can interpret each elements as an operation on a word. For example, if we look at the permutations of a four-letter word, the sub-group of permutations generated by the interchange of the first two or the last two letters is isomorphic to the Klein group corresponding to the elements (the elements correspond to the identity, (12), (34) and (12)(34) in cycle- notation). From an occult standpoint this is of course a natural

interpretation of temura, which essentially deals with operations on different entities through the mediation of letter-permutations, not unlike how physicists study physical objects by studying the equations describing them.

This introduces an interesting question in the Cabala, what other permutation-subgroups of the tetragrammaton have magickal correspondences? There are 24 elements in the group. This makes it possible to have subgroups of order 2,3,4,6,8 and 12. The subgroups of order 2,3 and 4 have already been discussed. The sub-group of order 12 is known as the alternating group, and consists of all the even permutations of the tetragrammaton. Its interesting to note that 12 permutations of the tetragrammaton are used linked to the zodiac and 12 tribes of Israel [1], but they are unfortunately not purely the even permutations.

The zodiac quite obviously corresponds to the cartesian product of a trinity (the alchemical elements) and the four elements (the klein group, which is $Z_2 * Z_2$). The result is $Z_3 * V = Z_{12}$, a quite neat cyclic group which fits well with the cyclical nature of the zodiac. Various elements generate the whole group or subgroups (like the quadruplicities and trinities).

However, this essay would become far too long if I elaborated on the interesting details of this (and more complex groups). Instead I encourage the reader to study and contemplate the meanings of the various mathematical concepts I have described, and find new structures. Mathematics is filled with fascinating objects, and their subtle interrelations and symbolism is an source of endless beauty.

References:

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